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THE TEMPERATURE FIELD OF A MEDIUM DUE TO AN INSTANTANEOUS CONCENTRATED FORCE

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The temperature field of an isotropic medium subjected to the action of an instantaneous concentrated force has been found. The temperature is determined as a solution of a coupled system of differential equations in an infinite space. A numerical analysis of the results obtained is made.

Introduction. The field of stresses generated in an infinite thermoelastic space due to the action of an instantaneous concentrated thermal source has been studied by the approximate small-parameter method [1]. The effect of a concentrated thermal source on an infinite thermoelastic space was also studied by the method of small time quantities [1]. The Green function for a concentrated aperiodic force for a quasistatic problem was obtained by V. Novatskii [2]. However, the problem of the thermal effect of an instantaneous concentrated force in an infinite thermoelastic medium remains unresolved. Such a problem is a constituent of the problem of constructing a fundamental solution of a coupled system of the differential equations of the dynamic thermoelasticity problem. The fundamental solution serves as a basis for constructing the potentials of a simple double layer and volume, which is the method for solving the dynamic problem of thermoelasticity of isotropic bodies with rough inclusions. Below, the temperature field induced in a medium by an instantaneous concentrated force is being studied.

Statement of the Problem. Let, in an infinite medium at the point $x = (x_1, x_2, x_3)$, an instantaneous concentrated force $X_i = \delta_{ik}\delta(x_1)\delta(x_2)\delta(x_3)\delta(t)$ be applied along the x_k axis. The temperature field induced by such a force factor is defined by a coupled system of differential equations ([1], p. 24):

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\,\text{grad div}\,\mathbf{u} - \rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \gamma\,\text{grad}\,T = -\mathbf{X}\,, \quad \nabla^2 T - \left(\frac{1}{\kappa}\right)\frac{\partial T}{\partial t} - \eta\,\text{div}\left(\frac{\partial\mathbf{u}}{\partial t}\right) = 0 \tag{1}$$

with the initial conditions

$$\mathbf{u}(x, 0) = 0\,, \quad \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{t=0} = 0\,, \tag{2}$$

$$T(x, t)\,\big|_{t=0} = 0\,, \tag{3}$$

where $\lambda = \dfrac{2\nu\mu}{1 - 2\nu}$; $\gamma = (3\lambda + 2\mu)\alpha_T$; $\kappa = \dfrac{\lambda_0}{c_\varepsilon}$.

Solution of the System. To solve system (1), we will represent the displacement and temperature in the form

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2\,, \tag{4}$$

$$T = T_1 + T_2\,. \tag{5}$$

The components of such representations are defined by the following systems of equations:

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$$L[\mathbf{u}_1] = -\mathbf{X}\,, \quad M[T_1] = 0 \tag{6}$$

and

$$L[\mathbf{u}_2] - \gamma \operatorname{grad} T_2 = \gamma \operatorname{grad} T_1\,, \quad M[T_2] - \eta \frac{\partial}{\partial t} \operatorname{div} \mathbf{u}_2 = \eta \operatorname{div} \mathbf{u}_1\,, \tag{7}$$

where the operators function according to the rules

$$L[\mathbf{u}] = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} \right), \quad M[T] = \nabla^2 T - \left(\frac{1}{a^2} \right) \left(\frac{\partial T}{\partial t} \right), \quad a^2 = \kappa\,.$$

In the Laplace integral transforms the solution of the first equation of (6) has the form [3]

$$U_i^{(k)}(x, p) = \frac{1}{4\pi\rho c_2^2} \left(\delta_{ik} \frac{1}{r} \exp\left(-\frac{p\tau}{c_1} \right) + \frac{c_2^2}{p^2} \frac{\partial^2}{\partial x_i \partial x_k} \left(\frac{1}{r} \left(\exp\left(-\frac{p\tau}{c_1} \right) - \exp\left(-\frac{p\tau}{c_2} \right) \right) \right) \right), \tag{8}$$

The inverse transform for (8) is the function [3]

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}\,, \quad c_2 = \sqrt{\frac{\mu}{\rho}}\,, \quad r = |x - y|\,.$$

$$U_{ik}(x, y, t) = \delta\left(1 - \frac{r}{c_1}\right) U_{ik}^{(1)}(x, y) + \delta\left(t - \frac{r}{c_2}\right) U_{ik}^{(2)}(x, y) + t \left[H\left(t - \frac{r}{c_1}\right) - H\left(t - \frac{r}{c_2}\right) \right] \frac{1}{r^2} U_{ik}^{(3)}(x, y)\,,$$

$$U_{ik}^{(1)}(x, y) = \frac{1}{4\pi\rho c_1^2} \frac{(x_i - y_i)(x_k - y_k)}{r^3}\,, \quad U_{ik}^{(2)}(x, y) = \frac{1}{4\pi\rho c_2^2} \left[\frac{\delta_{ik}}{r} \frac{(x_i - y_i)(x_k - y_k)}{r^3} \right], \tag{9}$$

$$U_{ik}^{(3)}(x, y) = -\frac{1}{4\pi\rho} \left[\frac{\delta_{ik}}{r} - 3 \frac{(x_i - y_i)(x_k - y_k)}{r^3} \right].$$

The solution of the second equation of (7) with a homogeneous initial condition is trivial [2]:

$$T_1 = 0\,.$$

We shall go over to the solution of system (7) which in an expanded form can be represented as

$$c_1^2 \operatorname{grad} \operatorname{div} \mathbf{u}_2 - c_2^2 \operatorname{rotrot} \mathbf{u}_2 - \frac{\partial^2 \mathbf{u}_2}{\partial t^2} - \frac{\gamma}{\rho} \operatorname{grad} T_2 = \frac{\gamma}{\rho} \operatorname{grad} T_1\,,$$

$$\nabla^2 T_2 - \frac{1}{a^2} \frac{\partial T_2}{\partial t} - \eta \frac{\partial}{\partial t} \operatorname{div} \mathbf{u}_2 = \eta \frac{\partial}{\partial t} \operatorname{div} \mathbf{u}_1\,. \tag{10}$$

Following [1], we assume that

$$\mathbf{u}_2 = \operatorname{grad} \mathbf{\Phi}_2\,. \tag{11}$$

Substituting (11) into (10), we obtain the equations

$$\left(\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) T_2 - \eta\nabla^2\left(\frac{\partial\Phi_2}{\partial t}\right) = \eta\,\frac{\partial}{\partial t}\operatorname{div}\mathbf{u}_1\,, \tag{12}$$

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi_2 - mT_2 = -\frac{1}{c_1^2}\frac{\gamma}{\rho}T_2\,, \tag{13}$$

We exclude the temperature T_2 from Eqs. (12) and (13). For this purpose we apply the operator $\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}$ to Eq. (12), multiply Eq. (13) by m, and add up the results. This will yield

$$\left(\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right)\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi_2 - m\eta\nabla^2\left(\frac{\partial\Phi_2}{\partial t}\right) = m\eta\,\frac{\partial}{\partial t}\operatorname{div}\mathbf{u}_1\,. \tag{14}$$

We solve Eq. (14). We apply the Laplace integral transform to it and represent it in the form ([1], p. 191)

$$\left(\nabla^2 - \lambda_1^2\right)\left(\nabla^2 - \lambda_2^2\right)\overline{\Phi}_2 = \frac{\varepsilon}{\kappa}\left(p\operatorname{div}\mathbf{u}_1 - (\operatorname{div}\mathbf{u}_1)\big|_{t=0}\right), \tag{15}$$

where $\lambda_1^2 + \lambda_2^2 = \frac{p}{\kappa}(1+\varepsilon)$, $\lambda_1^2\lambda_2^2 = \frac{p^3}{\kappa c_1^2}$, $\varepsilon = m\eta\kappa$.

Subject to Eq. (9), Eq. (15) takes the form

$$\left(\nabla^2 - \lambda_1^2\right)\left(\nabla^2 - \lambda_2^2\right)\overline{\Phi}_2 = A\,\frac{\partial}{\partial x_k}\left(\frac{1}{r}\exp\left(-p\,\frac{r}{c_1}\right)\right), \quad A = \frac{1}{4\pi\rho c_1^2}\,. \tag{16}$$

Similarly to [1], the unknown function $\overline{\Phi}_2(p)$ is sought in the form

$$\overline{\Phi}_2 = \frac{\partial}{\partial x_k}\overline{\Phi}_2^0\,. \tag{17}$$

As a result, we obtain the equation

$$\left(\nabla^2 - \lambda_1^2\right)\left(\nabla^2 - \lambda_2^2\right)\overline{\Phi}_2^0 = A\,\frac{1}{r}\exp\left(-\frac{pr}{c_1}\right),$$

$$\lambda_{1,2}^2 = \frac{1}{2}\left\{\frac{p\,(1+\varepsilon)}{\kappa} + \frac{p^2}{c_1^2} \pm \left[\left(\frac{p}{\kappa}(1+\varepsilon) + \frac{p^2}{c_1^2}\right)^2 - \frac{4p^3}{\kappa c_1^2}\right]^{1/2}\right\}, \quad \varepsilon = m\eta\kappa\,,$$

whose solution has the form

$$\overline{\Phi}_2^0 = -E\,\frac{1}{r}\exp(-\lambda_2 r) + E\,\frac{1}{r}\exp\left(-\frac{r}{c_1}p\right), \quad E = \frac{A}{\left(\frac{p^2}{c_1^2} - \lambda_1^2\right)\left(\frac{p^2}{c_1^2} - \lambda_2^2\right)} \tag{18}$$

or

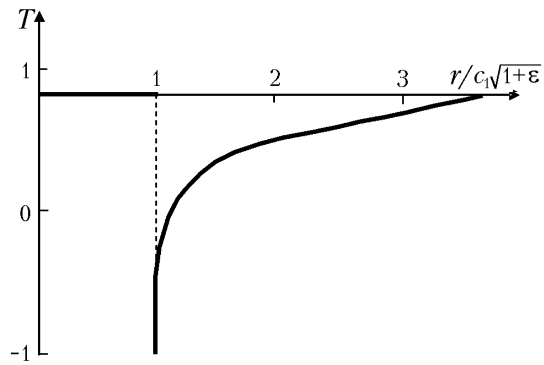

Fig. 1. Change in the temperature $T(r, t)$ for a constant value of time $t = 1$ depending on the relative value $\frac{r}{c_1\sqrt{1+\varepsilon}}$.

$$\overline{\Phi}_2^0 = \frac{A\kappa c_1^2}{\varepsilon p^3 r}\left(\exp\left(-\frac{pr}{c_1}\right) - \exp(\lambda_2 r)\right). \tag{19}$$

Recovering the inverted transform according to the Laplace inversion formula ([4], p. 499], we obtain

$$\Phi_2^0(r, t) = \frac{A\kappa c_1^2}{2}\frac{1}{r}\left\{\frac{1}{2}\left(t-\frac{r}{c_1}\right)^2 H\left(t-\frac{r}{c_1}\right) - \left[\left(t-\frac{r}{c_1\sqrt{1+\varepsilon}}\right)^2 - a_4 r\right] H\left(t-\frac{r}{c_1\sqrt{1+\varepsilon}}\right)\right\}, \tag{20}$$

where

$$a_4 = -\left.\frac{d^2\lambda^*}{dp^2}\right|_{p=0} = \frac{\sqrt{\kappa(1+\varepsilon)} + c_1^2(1+\varepsilon) - 4}{c_1^2\sqrt{\kappa(1+\varepsilon)}}; \quad \lambda^* = \lambda_2(p) - \frac{p}{c_1\sqrt{1+\varepsilon}}.$$

The temperature field $T_2(x, t)$ will be found according to Eq. (13):

$$T_2(x, t) = \frac{1}{m}\left[\frac{\gamma}{c_1^2 p} T_1 - \left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\Phi_2\right].$$

As a result, the temperature $T(r, t)$ in an explicit form is given by the expression

$$T(r, t) = -\frac{A\kappa(2+\varepsilon)}{1+\varepsilon}\frac{x_k - y_k}{r^3} H\left(t-\frac{r}{c_1\sqrt{1+\varepsilon}}\right) + \frac{A\kappa(2+\varepsilon)}{c_1(1+\varepsilon)^{3/2}}\delta\left(t-\frac{r}{c_1\sqrt{1+\varepsilon}}\right)\frac{x_k - y_k}{r}$$

$$+\frac{\varepsilon a_4}{c_1^3(1+\varepsilon)^{3/2}}\delta''\left(t-\frac{r}{c_1\sqrt{1+\varepsilon}}\right)\frac{x_k - y_k}{r} + \frac{2a_4}{c_1^3(1+\varepsilon)^{3/2}t}\delta\left(t-\frac{r}{c_1\sqrt{1+\varepsilon}}\right)\frac{x_k - y_k}{r}. \tag{21}$$

As adopted in [4], the symbol δ'' denotes the second derivative of the generalized Dirac delta function.

Discussion of Results. The relation obtained allows us to make the following inferences. In the limiting cases we have

$$T(r, t) \to 0 \quad \text{for} \quad r \to 0, \quad T(r, t) \to 0 \quad \text{for} \quad r \to \infty.$$

In the direction of action of a concentrated force, the disturbed temperature field at the continuity points changes as the value $O(1/r^2)$, where r is the distance from the point of application of the concentrated force, when compared to changes in other directions (of the order of $O(1/r^3)$).

Figure 1 demonstrates the change in the temperature $T(r, t)$ for a constant value of time $t = 1$ with change in the value of $\frac{r}{c_1\sqrt{1+\varepsilon}}$.

Conclusions. The expression found for the temperature field (21) is an integral part of the fundamental solution of the system of partial differential equations (1).

NOTATION

A, coefficient of Eq. (16); a_4, constant in the expression of the auxiliary function $\Phi_2^0(r, t)$; c_ε, specific heat in constant deformation; c_1 and c_2, velocities of elastic waves of longitudinal and transverse displacements, m/sec; $H(...)$, Heaviside unit function; p, complex number in the Laplace transform; $r = |x - y|$, distance between the points x and y, m; T, temperature of the medium, ^{o}C; T_1 and T_2, auxiliary temperature fields; t, time, sec; $\mathbf{u}$, displacement of the points of the medium; $\mathbf{u}_1$ and $\mathbf{u}_2$, components of the displacement $\mathbf{u}$; $\mathbf{X}$, mass forces; x and y, points of the space; α_T, coefficient of linear thermal expansion, $1/^o$C; δ_{ik}, Kronecker symbol; $\delta(...)$, Dirac delta function; ε, quantity depending on the properties of thermoelastic medium; κ, quantity defined by the thermal properties of the medium; λ_0, thermal conductivity, kcal/(cm·^{o}C); $\lambda_{1,2}$, roots of the governing characteristic equation; λ^*, Laplace transform of the auxiliary quantity; μ, shear modulus, kg/cm^2; ν, Poisson coefficient; ρ, density, kg/m^3; Φ_2 and $\overline{\Phi}_2$, scalar potential of the vector of displacements $\mathbf{u}_2$ and its transform; $\overline{\Phi}_2^0$, auxiliary function. Subscripts: i, $k = \overline{1, 3}$, coordinates of the points of a three-dimensional space, as well as components of the displacement matrix; ε, value measured in the case of constant deformation.

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